PROPAGATION OF UNSTEADY WEAK SHOCK WAVES IN A VIBRATIONALLY NONEQUILIBRIUM GAS SUBJECTED TO EXTERNAL RADIATION

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UDC 534.2.532

Acoustic waves can be amplified in a gas that has a nonequilibrium energy distribution among its internal degrees of freedom and is subjected to chemical reactions, external irradiation, electrical discharges, etc. [1-4]. This phenomenon magnifies the role of nonlinear effects and leads to the formation of shock waves [5-7]. The generation and propagation of shock waves are of major importance, because their occurrence in the active medium of gas lasers can cut off emission [8] and significantly lower the efficiency of chemical reactions in plasmatrons [9].

Here we investigate aspects of the propagation of unsteady weak disturbances in a gas when energy is transferred into the molecular vibrational degrees of freedom; we also determine the conditions for generation of a shock wave and the laws governing its evolution with time. We devote special attention to the important practical case of variable background. We obtain a solution to the problem of the evolution of a steady-state nonlinear disturbance in supersonic flow of a vibrationally nonequilibrium gas when energy is transferred into internal degrees of freedom in a layer of finite width. We investigate short waves, whose interaction time with the gas particles is much smaller than the characteristic relaxation time ("quasifrozen" approximation). Long waves, for which the presence of a relaxation process is equivalent to additional bulk viscosity and the basic mathematical apparatus is the Burgers equation ("quasiequilibrium" approximation) has been studied previously [7, 10].

One-dimensional gas flows with energy input into the vibrational degrees of freedom are described by the system of equations

$$\frac{d\rho}{dt}+\rho\frac{\partial u}{\partial x}+\frac{\nu\rho u}{x}=0,\quad \frac{du}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0,\quad \frac{dp}{dt}-a^2\frac{d\rho}{dt}=\rho(\gamma-1)F,$$
$$\frac{de_V}{dt}=I-F,\quad F=\varkappa I-\omega\left(e_V^*-e_V\right),\quad 0\leqslant\varkappa\leqslant 1,\quad \frac{d}{dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}.$$

Here p, ρ, and u are the pressure, density, and velocity of the gas, $a^2 = \gamma p/\rho$, γ is the "frozen" adiabatic index, e_V is the specific energy of the vibrational degrees of freedom, ν = 0.1 for planar axisymmetrical gas flows, I = I(t) is the specific power of the external energy source created by, for example, irradiation, and κ = const. We use the relations corresponding to a harmonic oscillator [11] for the equilibrium vibrational energy e_V^* and the reciprocal relaxation time ω:

$$e_V^*=\theta_V R/(\exp(\theta_V/T)-1),\quad \omega=k_1 p\exp\left(-k_2 T^{-1/3}\right) \tag{1}$$

(R is the gas constant, T is the temperature of translational degrees of freedom, θ_V is a characteristic vibrational temperature, and k_1 and k_2 are dimensioned positive constants, which depend on the properties of the gas and whose specific values may be found in [11]).

We consider the propagation of an acoustic pulse having an arbitrary waveform in a nonmoving homogeneous gas. The state of the gas changes with time under the influence of external radiation according to the system

$$\frac{\partial p^0}{\partial t}=\rho^0(\gamma-1)F^0,\quad e_V^0=e_{V0}^0-\frac{p^0-p_0^0}{(\gamma-1)\rho^0}+\int_0^t I(\xi)\,d\xi. \tag{2}$$

The superscript zero is used everywhere to indicate parameters describing the unperturbed state of the gas (background), and the subscript zero refers to data at t = 0. We rewrite the basic system of equations in the characteristic form

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 26-35, September-October, 1990. Original article submitted January 26, 1989; revision submitted April 18, 1989.

$$\frac{d_{1,2}}{dt}\left(u \pm \int \frac{dp}{\rho a}\right) = \pm(\gamma-1)\frac{F}{a} \mp \frac{\nu a u}{x}, \quad \frac{d_{1,2}x}{dt} = u \pm a,$$
$$\frac{d_3}{dt}\left(p - \int a^2 d\rho\right) = \rho(\gamma-1)F, \quad \frac{d_3}{dt}e_V = I - F, \quad \frac{d_3 x}{dt} = u. \tag{3}$$

We seek a solution in the form $u = u'(x, t)$, $p = p^0(t) = p'(x, t)$, ... (the prime refers to perturbations of the parameters of the gas). We assume that the wave disturbance is short, so that the same relations are applicable in the perturbed zone as at a weak shock wave [12, 13]:

$$p' = \rho^0 a^0 u', \quad a^0\rho' = \rho^0 u', \quad e'_V = 0. \tag{4}$$

The approximation (4) has the physical significance that the influence of reflected waves generated by a transmitted short wave, i.e., the acoustical second family, the entropy wave, and the wave associated with the excitation of internal degrees of freedom, can be disregarded. Indeed, the amplitudes of the reflected waves are proportional to their transmission times through the disturbed zone, i.e., to the respective quantities $\lambda/2a^0$ and λ/a^0 (λ is the wavelength). Linearizing Eqs. (3), we readily estimate the corrections of order λ to Eqs. (4) and the terms of order associated with these corrections in the first equation of the system (3) when the upper sign is chosen in it (see [4]). The requirement that the corrections associated with the finiteness of λ in comparison with terms not containing λ must be small yields a system of inequalities, which can be regarded as the criterion of "shortness" of the wave.

Dropping the superscript zero, we obtain

$$\left|\left[\frac{\gamma-2}{2}F + \gamma\omega(e_V^* - e_V) - \frac{\nu a^3}{x(\gamma-1)}\right]\left[(\gamma-1)\varkappa I - \frac{\gamma-2}{2}F + A\right]\frac{\gamma}{a^2} + \right.$$
$$\left. + \omega\left(A - \frac{\gamma-2}{2}F\right)\right| \ll \frac{a}{\lambda}\left|A + \frac{\nu a^3}{x(\gamma-1)}\right|, \quad \left|\frac{\gamma-1}{a^2}A + \frac{\nu a}{x}\right| \ll \frac{4a}{\lambda},$$
$$\left|\frac{\gamma}{2}F + A\right| \ll \frac{a^3}{\gamma\lambda}, \tag{5}$$
$$A(t) = \omega\left\{(e_V^* - e_V)\left[\gamma + (\gamma-1)\frac{k_2}{3T^{1/3}}\right] + \gamma(\gamma-1)\left(\frac{e_V^*}{a}\right)^2 \exp\frac{\theta_V}{T}\right\} + \frac{\gamma-2}{2}F.$$

Substituting Eq. (4) in the first equation (3), in which the upper sign is chosen, and integrating, we have

$$u'(x_0, t) = u'_0(x_0)\left(\frac{x_0}{x^0}\right)^{\gamma/2}\varphi(t), \quad x^0(x_0, t) = x_0 + \int_0^t a^0(\xi)\,d\xi,$$
$$\varphi(t) = \exp\left(-\int_0^t \frac{\gamma-1}{2a^{02}}A(\xi)\,d\xi\right). \tag{6}$$

The solution is written in parametric form; $x^0(x_0, t)$ gives the acoustical characteristics of the first family, which correspond to the background; $u_0'(x_0) = u'(x_0, 0)$. The velocity u' of the gas in the wave remains small under the condition $\varphi(t)(x_0/x^0)^{\nu/2} \sim 1$. If the inequality $A(t) \geq 0$ $(t > 0)$ holds, the disturbances do not grow. This inequality, which restricts the background parameters, can be regarded as the sufficient condition for stability of the initial state of the gas under disturbances of the given type. The derivative $dt/d\varphi$ has the significance of the characteristic decay time of a plane wave.

We now take into account the nonlinear effects associated with wave propagation, retaining first-order small terms in the equation for the acoustical characteristics:

$$dx/dt = a^0 + (\gamma+1)u'/2. \tag{7}$$

Substituting Eq. (6) in (7) and integrating, we obtain

$$x(x_0, t) = x_0 + \int_0^t a^0(\xi)\,d\xi + u'_0(x_0)\,\mathrm{X}(x_0, t),$$

$$X = \frac{\gamma+1}{2}\int_0^t \left(\frac{x_0}{x^0(x_0,\xi)}\right)^{\nu/2}\varphi(\xi)\,d\xi. \quad (8)$$

The equation for the envelope of the family of characteristics (8) $\partial x/\partial x_0 = 0$ can be written in the form

$$1 + (du_0'/dx_0)\,X(x_0, t) = 0 \quad (9)$$

(in the case of a cylindrical wave it is assumed that the characteristic wavelength is smaller than the distance from the wave front to the symmetry axis). The minimum value of t that satisfies Eq. (9) corresponds to the time of shock initiation, and the corresponding value of x_0 corresponds to the initial point of the characteristic at which the shock wave is generated.

It has been shown [5-7] that a compression disturbance propagating in a vibrationally nonequilibrium gas does not generate a shock wave in every case. In fact, if the integral X introduced in Eq. (8) converges in the limit $t \to \infty$, the formation of a shock wave requires that the initial profile $u_0'(x_0)$ contain intervals of large decay:

$$\frac{du_0'}{dx_0} < -\lim_{t\to\infty}\frac{1}{X}. \quad (10)$$

If the initial profile does not have such intervals, a shock wave is not generated. When the integral X diverges in the limit $t \to \infty$, any compression wave generates a shock wave.

It must be emphasized that when the behavior of the function X(t) [or $\Phi(t)$] is specified beforehand, it is always possible to choose a corresponding behavior of the power of the external radiation I = I(t). Thus, the specification of the function A(t), in terms of which X and Φ are expressed, with Eqs. (3) taken into account, leads to a system of two ordinary differential equations in $p^0(t)$ and an integral of I(t) that satisfies the conditions of the existence and uniqueness theorem.

The path of the shock wave crosses different characteristics at different times. The set of their initial points x_0 can be regarded as a Lagrangian coordinate system associated with the characteristics. If the initial point x_0 of the characteristic crossing the path of the shock wave at time t is indicated for every such t, the law of motion $x_0(t)$ of the shock wave is obtained in the coordinate system associated with the characteristics. The replacement of x_0 by the law $x_0 = x_0(t)$ in Eqs. (6) and (7) makes it possible to find the instantaneous amplitude of the discontinuity [u'](t) and the law governing its motion $x = x_S(t)$. The function $x_0(t)$ is derived from the equation

$$(\partial x/\partial x_0)/(dx_0/dt) + \partial x/\partial t = D. \quad (11)$$

Here D is the shock wave velocity, which is equal to half the sum of the characteristic velocities before and after the discontinuity [15], $\partial x/\partial t$ is given by Eq. (7), and $\partial x/\partial x_0$ is given by Eq. (8). Substituting the expressions for D, $\partial x/\partial t$, and $\partial x/\partial x_0$ in Eq. (11) and multiplying by $u_0'(x_0)$, within the assumed error limits we obtain an equation in total differentials, whose solution has the form

$$2\int_{x_0}^{x_{02}} u_0'(\xi)\,d\xi = X(x_0, t)\,u_0'^2(x_0). \quad (12)$$

Equation (12) gives the law of motion $x_0(t)$ of the shock wave in explicit form in the coordinate system associated with the characteristics; $x_{02} = x_0(0)$. The specific form of the function $x_0(t)$ can be determined by specifying at the initial time the waveform of the disturbance $u_0'(x_0)$, at which constraints do not accumulate in Eq. (12).

As an example, we consider a disturbance that has a triangular waveform at t = 0: $u_0' = k(x_0 - x_{01})$, $x_0 \le x_{02}$, k = const (Fig. 1); a discontinuity of amplitude $u_0'(x_{12})$ exists at the point x_{02}. In the case $\nu = 1$ we assume that the wavelength $x_{02} - x_{01}$ is much smaller than the distance x_{01} from the symmetry axis, so that the dependence of x_0 on t can be ignored in the factor $(x_0/x^0)^{\nu/2}$ in Eqs. (6) and (8), and we can let $x_0 = x_{02}$ (or $x_0 = x_{01}$), whereupon X = X(t). Determining the function $x_0(t)$ from Eq. (12) and substituting it in Eqs. (6) and (8), we obtain

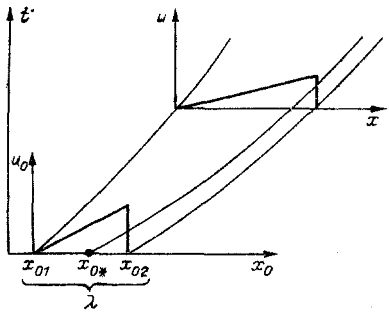

Fig. 1

$$x_s(t) = x_{01} + \int_0^t a^0(\xi)\,d\xi + \frac{x_{02} - x_{01}}{\sqrt{1 + kX(t)}} + \frac{u_0'(x_{02})\,X(t)}{\sqrt{1 + kX(t)}}\left[1 + O(u_0')\right]; \tag{13}$$

$$[u'](t) = \left(\frac{x_{02}}{x_s}\right)^{\nu/2} \frac{u_0'(x_{02})\,\varphi(t)}{\sqrt{1 + kX(t)}}\left[1 + O(u_0')\right]. \tag{14}$$

It is evident from Eqs. (12)-(14) that the convergence of the integral X in the limit $t \to \infty$ plays a decisive role. At the initial time, let $u_0'(x_0) > 0$ for $x_{01} < x_0 < x_{02}$, and let $u_0'(x_{01}) = 0$. If the integral X converges, Eq. (12) gives $x_0(t) \to x_{0*} > x_{01}$ in the limit $t \to \infty$. This means that the characteristic beginning at the point x_{0*} is an asymptote of the path of the shock wave. The asymptotic (for $t \to \infty$) shock decay law is given by Eq. (6) after the replacement of its right-hand side x_0 by x_{0*}. Consequently, to within a constant factor, the shock wave behaves like an acoustic (linear) wave at large t. The specific value of x_{0*} can be obtained by passing to the limit $t \to \infty$ in Eq. (12). For a triangular waveform

$$x_{0*} = x_{01} + (x_{02} - x_{01})/\sqrt{1 + kX(\infty)}. \tag{15}$$

If, on the other hand, the integral X diverges in the limit $t \to \infty$, none of the characteristics can serve as an asymptote for the path of the shock wave, and the shock decay law differs qualitatively from the acoustic law (6). The length of the waveform of the disturbance next to the discontinuity grows without bound, so that Eqs. (4)-(15) are invalid at not too large t.

The theory developed above can be used to describe the evolution of a periodic plane wave if its half-period λ satisfies the system (5). We note that the length of the half-period of such a wave is invariant [16], whereas the length of a solitary wave increases with time. If Eq. (10) holds at $t = 0$, discontinuities are subsequently formed in a periodic wave, which evolves into a sawtooth. In this case the law governing the motion of the shock wave in the coordinate system associated with the characteristics $x_0(t)$ is determined from the equation $x_{02} - x_0 = u_0'(x_0)X(t)$. As a result, in place of Eqs. (13) and (14) we have

$$x_s(t) = x_{02} + \int_0^t a^0(\xi)\,d\xi, \quad [u'](t) = \frac{u_0'(x_{02})\,\varphi(t)}{1 + kX(t)}. \tag{16}$$

We now consider some specific problems, for which certain simplifications permit the system of equations for the background parameters (3) to be integrated, and the behavior of the integral X in the limit $t \to \infty$ can be determined.

a) Let $I = I_0 = \text{const}$. Substituting the second equation (3) in the first, we obtain a nonlinear differential equation in p, whose solution has the asymptotic form in the limit $t \to \infty$

$$p = \frac{\gamma - 1}{\gamma}\rho\left(I_0 t + e_{v_0} + \frac{R\theta_v}{2} + \frac{p_0}{\rho(\gamma - 1)} + O\left(\frac{1}{t}\right)\right). \tag{17}$$

Using Eq. (16), from Eqs. (2) and (3) we find

$$\omega \sim k_1 \rho I_0 ((\gamma - 1)/\gamma) t, \quad e_V^* - e_V \sim O(1/t),$$

i.e., the characteristic relaxation time decays in the limit $t \to \infty$ as $1/t$, so that the gas tends to equilibrium, and $e_V^* - e_V$ decays more rapidly than $1/t$. For the function $\varphi(t)$ describing the behavior of weak disturbances in the linear approximation we obtain the asymptotic representation

$$\varphi(t) \sim \exp\left[-\frac{(\gamma-1)^3}{2\gamma^2} k_1 \rho \left(I_0 \frac{t^2}{2} - \frac{3}{5}\left(\frac{\gamma R I_0^2}{\gamma-1}\right)^{1/3} t^{5/3} + O\left(t^{4/3}\right)\right)\right], \tag{18}$$

which differs significantly from the equilibrium case $I = 0$, in which $\varphi(t) \sim \exp(-t \cdot \text{const})$, $t \to \infty$. According to relation (18), the integral X characterizing the qualitative behavior of nonlinear disturbances converges in the limit $t \to \infty$ for both plane and cylindrical waves.

b) Consider the behavior of weak disturbances in a gas whose vibrational degrees of freedom are elevated to the excited state $(e_{V_0} - e_{V_0}^*)$ without external radiation $(I = 0)$. Relaxation subsequently causes the pressure and translational temperature to increase and the vibrational energy to decrease. We assume that T is high, θ_V/T is small, and in the first approximation $e_V^* = p/\rho$. This approximation is valid for a number of diatomic gases at high pressures, when a large difference exists between the characteristic vibrational temperature θ_V and the temperature at which dissociation processes are essential [11]. We can disregard the dependence of ω on $T^{1/3}$ and assume that $\omega = k_3 T$ and $k_3 = \text{const}$, provided that the range of T is not too great. The system (3) is integrable on the basis of these assumptions:

$$p(t) = p_0 \frac{1+\delta}{1+\delta\exp(-\alpha t)}, \quad \alpha = k_1 p_0 \gamma (1+\delta), \quad \delta = \frac{(\gamma-1)\left(e_{V_0} - e_{V_0}^*\right)}{\gamma e_{V_0}^*}. \tag{19}$$

Substituting Eq. (19) in (6), we have

$$\varphi(t) = \exp\left(-\frac{\alpha t}{2}\left(\frac{\gamma-1}{\gamma}\right)^2 + \frac{3\gamma^2 - 2\gamma + 2}{4\gamma^2} \ln\left(\frac{1+\delta}{1+\delta\exp(-\alpha t)}\right)\right). \tag{20}$$

The first factor in Eq. (20) is an increasing function of time, and the second is a decreasing function. The function $\varphi(t)$ decreases with time if $\delta \le \delta_m = 2(\gamma - 1)^2/\gamma(\gamma + 2)$; for $\delta > \delta_m$ it increases up to a time $t = t_m$, which is given by the expression

$$t_m = \frac{1}{\gamma(1+\delta)\omega_0} \ln\left(\delta \frac{\gamma(\gamma+2)}{2(\gamma-1)^2}\right),$$

and then decreases, tending to zero in the limit $t \to \infty$. Bauer and others [1-4] have previously mentioned the possibility of the amplification of harmonic waves propagating in a homogeneous gas with fixed parameters, which sustains nonequilibrium at a constant level through the balanced transfer of energy into internal degrees of freedom and by outgoing heat transfer.

Substituting t_m in Eq. (20) and expanding $\varphi(t_m)$ in a series with respect to the small parameter $(\gamma - 1)^2/2\gamma^2$, we obtain

$$\varphi(t_m) = 1 + \left(\frac{\delta}{\delta_m} - 1\right)\frac{(\gamma-1)^2}{2\gamma^2} + O\left(\frac{(\gamma-1)^2}{2\gamma^2}\right). \tag{21}$$

The requirement that the second term on the right-hand side of Eq. (21) must be small in comparison with unity can be regarded as a criterion of linear stability of the state of a vibrationally excited gas under acoustic disturbances. This criterion states that the initial relative nonequilibrium must be small:

$$\frac{e_{V_0} - e_{V_0}^*}{e_{V_0}^*} \ll \frac{4\gamma^2}{(\gamma-1)(2+\gamma)}\left[1 + O\left(\frac{(\gamma-1)^2}{2\gamma^2}\right)\right].$$

For nonlinear disturbances with a shock wave, according to Eq. (20), the integral X converges for both plane and cylindrical waves, so that the path of the shock wave has a characteristic as its asymptote, and the length of the waveform adjacent to the discontinuity is bounded in the limit $t \to \infty$. We emphasize that growth of the disturbances leads to the magnification of nonlinear effects and rapid shock generation, whereupon nonlinear theory must be invoked in order to describe the evolution of the disturbance. For example, whereas the amplitude of a plane wave increases directly as t in the linear case [$\varphi(t) \sim t$ at $t > 0$], in nonlinear theory, according to Eq. (14), the amplitude of a plane wave with a triangular waveform remains small and, according to Eq. (16), the amplitude of a sawtooth decays as $1/t$.

We now consider steady supersonic planar flows of a gas that admits the excitation of vibrational degrees of freedom; such processes differ very little from one-dimensional flows. They can occur, for example, in flow around slender bodies with moderately supersonic velocity at a small angle of attack or in flow through ducts with slight wall roughness. We choose the coordinates x, y so that the direction of the x axis coincides with the freestream direction. We assume that the gas flow is exposed to external radiation of intensity $I(x)$ in some interval $x_1 < x < x_2$; $I(x) \equiv 0$ at $x \notin (x_1, x_2)$. We write the system of equations describing supersonic planar flow of a vibrationally relaxing gas in the characteristic form

$$v\frac{d_{\pm}u}{dx} - u\frac{d_{\pm}v}{dx} \mp \frac{\sqrt{M^2-1}}{\rho}\frac{d_{\pm}p}{dx} = \frac{F(\gamma-1)}{u^2-a^2}\left(\mp u\sqrt{M^2-1} - v\right),$$
$$\frac{d_{\pm}y}{dx} = \frac{uv \pm a^2\sqrt{M^2-1}}{u^2-a^2}, \quad \frac{d_0}{dx}\left(\frac{u^2+v^2}{2}\right) + \frac{1}{\rho}\frac{d_0 p}{dx} = 0, \tag{22}$$
$$\frac{d_0 p}{dx} - a^2\frac{d_0\rho}{dx} = (\gamma-1)\frac{\rho F}{u}, \quad u\frac{d_0 e_V}{dx} = I - F, \quad \frac{d_0 y}{dx} = \frac{v}{u}$$

(u and v are the projections of the velocity onto the x and y axes, and M is the local Mach number; $v \equiv 0$ when disturbances are absent). The first equation of the system (22) is written along an acoustical characteristic of the first or second family, which is described by the second equation (the upper sign corresponds to the first family, and the lower sign to the second family), and the next three equations are written along streamlines that obey the last equation.

We consider the linear problem of a steady-state supersonic flow disturbance concentrated in a narrow zone of width λ between two acoustical characteristics of the first family. If λ is sufficiently small, the disturbances obey the same relations as at a weak oblique shock [17]:

$$u' + v/\sqrt{M^{02}-1} = 0, \quad \rho^0 u^0 u' + p' = 0, \quad p' - a^{02}\rho' = 0, \quad e'_V = 0 \tag{23}$$

($v' \equiv v$, since $v^0 = 0$). In the linear approximation the wave front coincides with a characteristic of the acoustical family. The short-wave approximation has the significance that the discrepancies in Eqs. (23) are proportional to those segments of the characteristics of the system (22) (acoustical second family and streamline) which are contained in the perturbed zone, i.e., the discrepancies are proportional to λ and are small if λ is sufficiently small.

Linearizing the first equation of the system (22), in which the upper sign is chosen, and substituting Eq. (23) into the result, we obtain a linear differential equation, whose solution can be represented in a parametric form analogous to Eq. (6):

$$u'(x, y_0) = u'_0(y_0)\exp\left(-\int_0^x \frac{\gamma-1}{2a^{02}(\xi)}A_*(\xi)\,d\xi\right) \equiv u'_0(y_0)\varphi_*(x), \quad y = y^0(x, y_0) = y_0 + \int_0^x \frac{d\xi}{\sqrt{M^{02}(\xi)-1}}, \quad y^0(0, y_0) \equiv y_0,$$
$$A_*(x) = \frac{\omega^0 M^0}{a^0\left(M^{02}-1\right)}\left(\left(e_V^{*0} - e_V^0\right)\left[\frac{3\left(\gamma M^{02}+1\right)}{2\left(M^{02}-1\right)} + \frac{k_2(\gamma-1)}{3T^{01/3}}\right] + \right. \tag{24}$$
$$\left. + \gamma(\gamma-1)\left(\frac{e_V^{*0}}{a^0}\right)^2 \exp\frac{\theta_V}{T} - \frac{\varkappa I\left(\gamma M^{02}+2\gamma+3\right)}{2\omega^0\left(M^{02}-1\right)}\right).$$

The subscript zero is used everywhere to indicate the values of the parameters at x = 0. The acoustical characteristics of the first family, at which $u_0' \neq 0$, pass through points of the segment $[y_{01}, y_{02}]$ of the y axis.

In Eq. (24) the functions $M^0(x)$ and $A_*(x)$ are expressed in terms of the freestream parameters, which are given by the relations

$$\rho^0 u^0 = \rho_0 u_0,\ p^0 + \rho^0 u^{0\,2} = p_0 + \rho_0 u_0^2,\ u^0 \frac{de_V^0}{dx} = (1-\varkappa) I + \omega^0 \left(e_V^{*0} - e_V^0\right),$$

$$\left(e_V^0 - e_{V_0}\right) + \frac{1}{2}\left(u^{0\,2} - u_0^2\right) + \frac{a^{0\,2} - a_0^2}{\gamma - 1} = \int_0^x \frac{I(\xi)\, d\xi}{u^0(\xi)}. \tag{25}$$

The substitution $u^0 = u_0(1 + \Delta)$ reduces the system (25) to the equation

$$\frac{d\Delta}{dx} = \frac{(\gamma-1)\,\omega^0(\Delta)\left(e_V^{*0}(\Delta) - e_V^0(\Delta)\right) - \varkappa I}{a_0^2 u^0(\Delta)\left(M_0^2 + M_0^2(\gamma+1)\Delta - 1\right)}. \tag{26}$$

If the denominator on the right-hand side of Eq. (26) is equal to zero, the instantaneous value of the freestream Mach number $M^0(x)$ becomes equal to unity. We assume that the flow is supersonic everywhere $[M^0(x) > 1]$ and, accordingly, that $\Delta > -(M_0{}^2 - 1)/((\gamma + 1)M_0{}^2)$; this assumption corresponds to not too large values of the energy input to the gas on the interval (x_1, x_2) and is valid in all cases of practical interest. Including first-order small terms in the acoustical characteristic equations, we have

$$\frac{dy}{dx} = \frac{1}{\sqrt{M^{0\,2} - 1}} - \frac{M^{0\,3}(\gamma+1)}{2a^0\left(M^{0\,2} - 1\right)^{3/2}}\, u'(x, y_0). \tag{27}$$

Substituting relations (23) in Eq. (27) and integrating, we obtain

$$y(x, y_0) = y^0(x, y_0) + u_0'(y_0)\, X_*(x),\ X_*(x) = -\frac{\gamma+1}{2}\int_0^x \frac{M^{0\,3}(\xi)\,\varphi(\xi)\, d\xi}{a^0(\xi)\left(M^{0\,2}(\xi) - 1\right)^{3/2}}. \tag{28}$$

A shock wave is generated at the point of intersection of the characteristics (28), which is determined from the condition $\partial y/\partial y_0 = 0$ or, rewriting it in a form similar to (9) on the basis of Eq. (28):

$$1 + \left(du_0'/dy_0\right) X_*(x) = 0. \tag{29}$$

Inasmuch as the gas is irradiated only in the interval $x_1 < x < x_2$, the state of the gas approaches equilibrium with increasing x, the function φ_* decays exponentially in the limit $x \to \infty$, and the integral X converges. In this connection, according to Eq. (29), a steady-state disturbance superimposed on supersonic flow contains a shock wave only when the initial waveform $u_0'(y_0)$ has intervals of large growth:

$$\frac{du_0'}{dy_0} > -\lim_{x\to\infty} \frac{1}{X_*} > 0. \tag{30}$$

If condition (30) is not statisfied, a shock wave does not exist. This implies, in particular, that supersonic flow of a vibrationally relaxing gas over a concave wall of small curvature is not accompanied by the formation of a shock wave (Fig. 2).

The x dependence of the shock wave amplitude is found in the same way as in the unsteady case. Specifically, the shock equation $y_0 = y_0(x)$ in the Lagrangian coordinate system y_0 associated with characteristics of the first family can be deduced from an equation analogous to Eq. (11):

$$(\partial y/\partial y_0)(dy_0/dx) + \partial y/\partial x = D_* \tag{31}$$

[$\partial y/\partial x$ and $\partial y/\partial y_0$ are determined from Eq. (28), and D_* is the half-sum of the characteristic velocities before and after the discontinuity]. We multiply Eq. (31) by $u_0'(y_0)$ and integrate:

$$2\int_{y_0}^{y_{02}} u_0'(\xi)\, d\xi = X_*(x)\, u_0'^{\,2}(y_0). \tag{32}$$

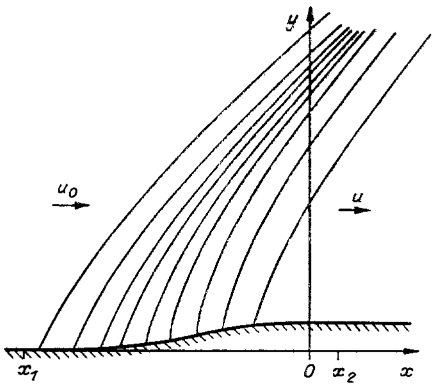

Fig. 2

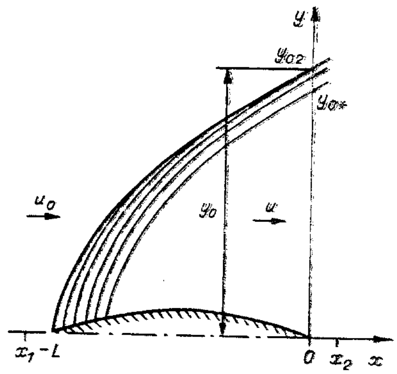

Fig. 3

It is assumed in the derivation of Eq. (32) that the preshock perturbations are equal to zero: $u_0' = 0$ at $y_0 > y_{02}$ (y_{02} is the coordinate of the shock wave at x = 0). Solving Eq. (32) for y_0 and substituting the result together with y_0 in Eq. (28) and in the first equation (24), we obtain the shock equation $y_s(x) = y(x, y_0(x))$ and the law governing its amplitude variation $[u'](x) = u_0'[y_0(x)]\varphi_*(x)$. Since the integral X_* converges in the limit $x \to \infty$, we infer from Eq. (32) that the shock wave has an acoustical characteristic of the first family as its asymptote; the initial point $y_{0*} = y_0(\infty) < y_{02}$ to the limit $x \to \infty$ and specifying beforehand the initial waveform $u_0'(y_0)$ of the disturbances.

We now discuss steady supersonic flow over a slender shape, which is symmetrical about the x axis and has an attached shock wave at its initial point; the perturbations induced by the shape in the flow are linear in y_0: $u_0'(y_0) = k_* y_0$, $y_0 \le y_{02}$ (Fig. 3). Solving Eq. (32) for y^0 and substituting the resulting function $y_0(x)$ in Eqs. (24) and (28), we have

$$[u'](x) = \frac{u_0'(y_{02})\,\varphi_*(x)}{\sqrt{1 + k_* X_*(x)}}, \quad y_s(x) = y^0(x, y_{02}) + \frac{u_0'(y_{02})\,X_*(x)}{\sqrt{1 + k_* X_*(x)}}. \tag{33}$$

As an example, we consider the special case in which I ≡ 0 at x > −L, a slender convex body is encountered in the cross section x = −L, vibrational degrees of freedom are excited, and the temperature and pressure of the gas are sufficiently high, so that the assumptions $e_v^* = p/\rho$ and $\omega = k_3 p$ can be made in the first approximation. Changing the variable of integration in Eq. (24) by means of Eq. (26), we then obtain

$$\ln \varphi_* = -\frac{M_0^2}{2}\int_0^{\Delta}\left(\frac{3(\gamma M_0^2 + 1)\,d\xi}{2\left(M_0^2 - 1 + (\gamma + 1) M_0^2\xi\right)\left(1 - \gamma M_0^2\xi\right)} + \right.$$
$$\left. + \frac{(\gamma - 1)^2 (1 + \xi)\,d\xi}{\xi\left(2\gamma - 1 - \gamma^2 M_0^2 - \frac{3\gamma - 1}{2}\gamma M_0^2\xi\right) - \gamma\delta}\right).$$

If $\delta \le \delta_m^* = 2(\gamma - 1)^2(M_0^2 - 1)/(3\gamma^2 M_0^2 + 3\gamma)$, then φ_* decays with increasing x, but if $\delta > \delta_m^*$, we find that φ_* increases to the value φ_{*m} and then begins to decrease. In this case

$$\varphi_{*m} = 1 + \left(\frac{\delta}{\delta_m} - 1\right)\frac{(\gamma - 1)^2 M_0^2}{2\left(\gamma^2 M_0^2 + 1 - 2\gamma\right)} + O((\gamma - 1)^4). \tag{34}$$

The smallness of the second term in Eq. (34) in comparison with the first term can be taken as a criterion of linear stability.

We note that a disturbance which contains discontinuities and grows from the standpoint of linear theory can decay in the nonlinear approximation. For example, according to Eqs. (33), the shock wave amplitude remains small for $\varphi_* \sim x$.

The authors are grateful to V. A. Levin for attention.

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